Ross Program 2019 Application Problems

This document is part of the application to the *Ross Mathematics Program*, and will remain posted at https://rossprogram.org/students/to-apply from January to April.

The Admission Committee will make admission decisions on a rolling basis, starting in March 2019. The deadline for applications is April 1, but spaces will fill as applications arrive. For adequate consideration of your application, it is best to **send in your solutions well before the end of March.**

Work independently on the problems below. We are interested in seeing how you approach unfamiliar math problems, not whether you can find answers by searching through web sites or books, or by asking other people.

Please submit your own work on each of these problems.

For each problem, explore the situation (with calculations, tables, pictures, etc.), observe patterns, make some guesses, test the truth of those guesses, and write logical proofs when possible. Where were you led by your experimenting?

Include your thoughts even though you may not have found a complete solution. If you've seen one of the problems before (e.g. in a class or online), please include a reference along with your solution.

We are not looking for quick answers written in minimal space. Instead, we hope to see evidence of your explorations, conjectures, and proofs written in a readable format.

> The quality of mathematical exposition, as well as the correctness and completeness of your solutions, are factors in admission decisions.

Please convert your problem solutions into a PDF file. You may type the solutions using LaTeX or a word processor, and convert the output to PDF format. Alternatively, you may scan or photograph your solutions from a handwritten paper copy, and convert the output to PDF. (Please use dark pencil or pen and write on only one side of the paper.)

Note: Unlike the problems here, each Ross Program course concentrates deeply on one subject. These problems are intended to assess your general mathematical background and interests.

Problem 1

What numbers can be expressed as an alternating-sum of an increasing sequence of powers of 2? To form such a sum, choose a subset of the sequence $1, 2, 4, 8, 16, 32, 64, \ldots$ (these are the powers of 2). List the numbers in that subset in increasing order (no repetitions allowed), and combine them with alternating plus and minus signs. For example,

$$\begin{array}{llll} 1 = -1 + 2; & 2 = -2 + 4; & 3 = 1 - 2 + 4; & \\ 4 = -4 + 8; & 5 = 1 - 4 + 8; & 6 = -2 + 8; & \text{etc.} \end{array}$$

Note: The expression $5 = -1 - 2 + 8$ is invalid because the signs are not alternating.

(a) Is every positive integer expressible in this fashion? If so, give a convincing proof.

(b) A number might have more than one expression of this type. For instance

$$3 = 1 - 2 + 4 \quad \text{and} \quad 3 = -1 + 4.$$

Given a number n, how many different ways are there to write n in this way? Prove that your answer is correct.

(c) Do other sequences (a_n) of integers have similar alternating-sum properties? Explore a sequence of your choice and make observations.

One idea: Can every integer k be expressed as an alternating sum of an increasing sequence of Fibonacci numbers? Can some integers be expressed as such sums in many different ways?

Or you could explore some other sequence instead.

Problem 2

A polynomial $f(x)$ has the *factor-square property* (or FSP) if $f(x)$ is a factor of $f(x^2)$. For instance, $g(x) = x - 1$ and $h(x) = x$ have FSP, but $k(x) = x + 2$ does not.

Reason: $x - 1$ is a factor of $x^2 - 1$, and x is a factor of x^2, but $x + 2$ is not a factor of $x^2 + 2$.

Multiplying by a nonzero constant "preserves" FSP, so we restrict attention to polynomials that are *monic* (i.e., have 1 as highest-degree coefficient).

What patterns do monic FSP polynomials satisfy?
To make progress on this topic, investigate the following questions and justify your answers.

(a) Are x and $x - 1$ the only monic FSP polynomials of degree 1?

(b) List all the monic FSP polynomials of degree 2.
To start, note that x^2, $x^2 - 1$, $x^2 - x$, and $x^2 + x + 1$ are on that list.
Some of them are products of FSP polynomials of smaller degree. For instance, x^2 and $x^2 - x$ arise from degree 1 cases. However, $x^2 - 1$ and $x^2 + x + 1$ are new, not expressible as a product of two smaller FSP polynomials.
Which terms in your list of degree 2 examples are new?

(c) List all the monic FSP polynomials of degree 3. Which of those are new?

Can you make a similar list in degree 4?

(d) Answers to the previous questions might depend on what coefficients are allowed. List the monic FSP polynomials of degree 3 that have integer coefficients. Separately list those (if any) with complex number coefficients that are not all integers.
Can you make similar lists for degree 4?
Are there examples of monic FSP polynomials with real number coefficients that are not all integers?

Problem 3

For a positive integer k, let S_k be the set of numbers $n > 1$ that are expressible as $n = kx + 1$ for some positive integer x. The set S_k is closed under multiplication. That is: If $a, b \in S_k$ then $ab \in S_k$.
Definition. Suppose $n \in S_k$. If n is expressible as $n = ab$ for some $a, b \in S_k$, then n is called k-composite. Otherwise n is called a k-prime.

For example, $S_4 = \{5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, \ldots\}$. The numbers 25, 45, 65, 81, ... are 4-composites, while 5, 9, 13, 17, 21, 29, ... are 4-primes.

1. Which $n \in S_4$ are 4-primes? (Answer in terms of the standard prime factorization of n.)
   Show: Every $n \in S_4$ is either a 4-prime or a product of some 4-primes.
   But "unique factorization into 4-primes" fails. To prove that, find some
   $n = p_1 p_2 \cdots p_s$ and $n = q_1 q_2 \cdots q_t$ where each p_j and q_k is a 4-prime, but the list $(q_1, \ldots, q_t)$ is not just a rearrangement of the list $(p_1, \ldots, p_r)$.

2. Which $n \in S_3$ are 3-primes? Is there unique factorization into 3-primes?

3. Suppose a positive integer k is given, along with its standard prime factorization.
   Which integers $n \in S_k$ are k-primes?
   For which k does the system S_k have unique factorization into k-primes?

   Prove that your answers are correct.

Problem 4

If $\mathcal{S}$ is a set of points in space, define its *line-closure*

$\mathcal{L}(S) =$ the union of all lines passing through two distinct points of $\mathcal{S}$.

That is: Point X lies in $\mathcal{L}(S)$ if there exist distinct points $A, B \in \mathcal{S}$ such that A, B, X are collinear. Then $\mathcal{S} \subseteq \mathcal{L}(\mathcal{S})$, provided $\mathcal{S}$ contains at least two points.

For example, if points A, B, C do not lie in a line, then $\mathcal{L}(\{A, B, C\})$ is the union of three lines whose intersection points are A, B, C. In this case, $\mathcal{L}(\mathcal{L}(\{A, B, C\}))$ is the whole plane containing those points.

1. When can a set $\mathcal{S}$ equal its own line-closure: $\mathcal{L}(\mathcal{S}) = \mathcal{S}$?
   Prove that your answer is correct.

2. Suppose A, B, C, D are points in 3-space that do not all lie in one plane. Describe the sets $\mathcal{L}(\{A, B, C, D\})$ and $\mathcal{L}(\mathcal{L}(\{A, B, C, D\}))$.

3. Suggest some ways that these ideas can be generalized.

We hope you enjoyed working on these problems! For more information about this summer math program visit https://rossprogram.org/. You may email your questions and comments to ross@rossprogram.org.