Ross Program 2017 Application Problems

This document is part of the application to the *Ross Mathematics Program*, and is posted at http://u.osu.edu/rossmath/.

The Admission Committee will start reading applications on March 1, 2017, making admission decisions on a rolling basis. The deadline for applications is April 1, but spaces will fill as applications arrive. For adequate consideration of your application, it is best to **submit your solutions well before the end of March.**

Each applicant should work independently on the problems below. We are interested in seeing how you approach unfamiliar math problems, not whether you can find answers by searching through web sites or books, or by asking experts.

Please submit your own work on all of these problems.

For each problem, explore the situation (with calculations, tables, pictures, etc.), observe patterns, make some guesses, test the truth of those conjectures, and describe the progress you have made. Where were you led by your experimenting?

Include your thoughts even though you may not have completely solved the problem. If you've seen one of the problems before (e.g. in a class or online), please include a reference along with your solution.

Please convert your problem solutions into a PDF file. That PDF file can be created by scanning your problem solutions from a handwritten paper copy. (It's best to use dark pencil or pen using only one side of the paper.) Alternatively, you could type up the solutions using a word processor or with LaTeX, and then convert the output to PDF format.

Note: each Ross Program course concentrates deeply on one subject, unlike the problems here. This Problem Set is an attempt to assess your general mathematical background and interests.

Problem 1

What numbers can be expressed as an alternating-sum of an increasing sequence of powers of 2? To form such a sum, choose a subset of the sequence $1, 2, 4, 8, 16, 32, 64, \ldots$ (these are the powers of 2). List the numbers in that subset in increasing order (no repetitions allowed), and combine them with alternating plus and minus signs. For example,

$$\begin{array}{llll} 1 = -1 + 2; & 2 = -2 + 4; & 3 = 1 - 2 + 4; & \\ 4 = -4 + 8; & 5 = 1 - 4 + 8; & 6 = -2 + 8; & \text{etc.} \end{array}$$

Note: the expression $5 = -1 - 2 + 8$ is invalid because the signs are not alternating.

(a) Is every positive integer expressible in this fashion? If so, give a convincing proof.

(b) A number might have more than one expression of this type. For instance

$$3 = 1 - 2 + 4 \quad \text{and} \quad 3 = -1 + 4.$$

Given a number n, how many different ways are there to write n in this way? Explain why your answer is correct.

Problem 2

Given a line segment from the origin O to a point P, a geometer constructs a rectangle of area 1 with the segment OP as base, and oriented counter-clockwise from the segment.

Starting from $P_0 = (1, 0)$ on the x-axis, she draws a square with base OP_0. (That square is red in the picture.) With "diagonal point" $P_1 = (1, 1)$, she draws an area 1 rectangle on base OP_1 (blue in the picture). With diagonal point $P_2 = (\frac{1}{2}, \frac{3}{2})$, she draws the next rectangle with base OP_2 (green).

Continue the process: Using diagonal point P_n of the n^{th} rectangle, construct the next rectangle with base OP_n, area 1, and counterclockwise from OP_n.

(a) Let Q_n be the fourth corner of the n^{th} rectangle. The picture indicates that Q_0 lies on segment Q_1P_2, and Q_1 lies on segment Q_2P_3.

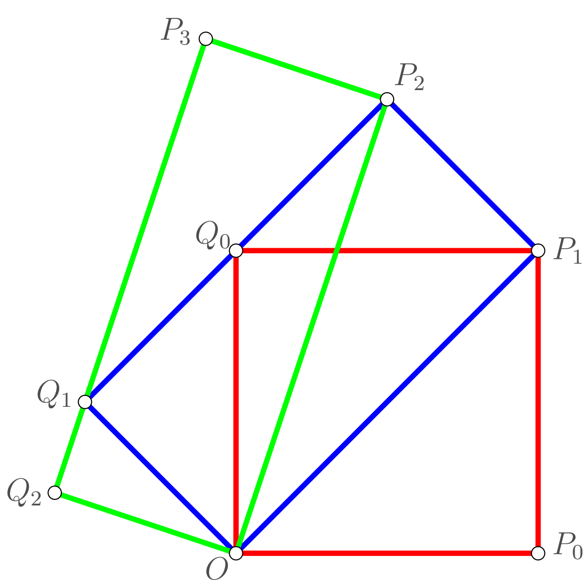
Does Q_n lie on the the segment $Q_{n+1}P_{n+2}$, for every $n = 0, 1, \ldots$?

If so, provide an explanation (proof) of why that happens.

(b) Let $b_n = |OP_n|$ be the base length of the n^{th} rectangle. Then the adjacent side has length $|OQ_n| = 1/b_n$, because the area is 1. For instance, $b_0 = 1,\ b_1 = \sqrt{2}$, ; and $b_2 = \sqrt{5/2}$.

Do the lengths b_n grow without bound as n increases?

Do the points P_n spiral repeatedly around O as n increases?



Problem 3

How many checkers can be placed on an 6×6 board without forming a rectangle? Restrict attention to those rectangles with horizontal and vertical sides.

More generally, define $R(m,n)$ to be the largest number of dots that can be placed within an $m \times n$ array without making a rectangle of dots (not counting tilted rectangles).

For instance, in a 3×3 array an L-shaped arrangement of 5 dots contains no rectangle. Therefore $R(3,3) \geq 5$. But 5 is not maximal: there is a rectangle-free arrangement of 6 dots, proving that $R(3,3) \geq 6$. After some work I managed to prove:

Every arrangement of 7 dots in a 3×3 array must include a rectangle.

This shows that $R(3,3) = 6$.

(a) Does $R(2,n) = n+1$ for every $n \geq 2$? Can you evaluate $R(3,4), R(3,5), \ldots$? How about $R(4,4)$?

(b) Investigate other values of $R(m,n)$, observing patterns, making conjectures, and constructing proofs.

Problem 4

A polynomial $f(x)$ has the *factor-square property* (or FSP) if $f(x)$ is a factor of $f(x^2)$. For instance, $g(x) = x - 1$ and $h(x) = x$ have FSP, but $k(x) = x + 2$ does not.

Reason: $x - 1$ is a factor of $x^2 - 1$, and x is a factor of x^2, but $x + 2$ is not a factor of $x^2 + 2$.

Multiplying by a nonzero constant "preserves" FSP, so we restrict attention to polynomials that are *monic* (i.e., have 1 as highest-degree coefficient).

What is the pattern to these FSP polynomials? To help you make progress on this general question, investigate the following questions and justify your answers.

(a) Are x and $x - 1$ the only monic polynomials of degree 1 with FSP?

(b) Check that x^2, $x^2 - 1$, $x^2 - x$, and $x^2 + x + 1$ all have FSP. Determine all the monic degree 2 polynomials with FSP.

(c) Some of our examples are products of FSP polynomials of smaller degree. For instance, x^2 and $x^2 - x$ come from degree 1 cases. However, $x^2 - 1$ and $x^2 - x + 1$ are new, not expressible as a product of two smaller FSP polynomials.

Are there monic FSP polynomials of degree 3 that are new (not built from FSP polynomials of smaller degree)? Are there such examples of degree 4?

(d) The examples written above all had integer coefficients. Do answers change if we allow polynomials whose coefficients are allowed to be any real numbers? Or if we allow polynomials whose coefficients are complex numbers?

Problem 5

Many students are familiar with Pascal's Triangle:

Row 0:	1								
Row 1:	1	1							
Row 2:	1	2	1						
Row 3:	1	3	3	1					
Row 4:	1	4	6	4	1				
Row 5:	1	5	10	10	5	1			
Row 6:	1	6	15	20	15	6	1		
Row 7:	1	7	21	35	35	21	7	1	
Row 8:	1	8	28	56	70	56	28	8	1
			etc.						

Standard notation: $\binom{n}{k}$ is the the k^{th} entry of Row n. (Here, counting starts at zero rather than one.) For example, $\binom{7}{2} = 21$ and $\binom{7}{3} = 35$. Each entry equals the sum of the entries directly above and above-left (except for the 1's at the ends of rows).

The entry $\binom{n}{k}$ is the number of combinations of n objects taken k at a time. That term also appears in the *Binomial Theorem*: $\binom{n}{k}$ equals the coefficient of x^k in the expansion of $(1+x)^n$.

The sum across Row n equals 2^n, and sums of every second entry of Row n are half of that total. For instance, when $n = 8$:

$$1+8+28+56+70+56+28+8+1 \;=\; 256 \;=\; 2^8.$$

$$\begin{aligned} 1+28+70+28+1 &= 128 &= 2^8/2, \\ 8+56+56+8 &= 128 &= 2^8/2. \end{aligned}$$

Several different methods lead to proofs of those results for every n. You might use induction, direct counting of combinations, or evaluating $(1+x)^n$ at $x = 1$ and $x = -1$.

(a) What about the sums of every third entry of Row n? In Row 8 those sums are:

$1+56+28 = 85 \qquad 8+70+8 = 86 \qquad 28+70+8 = 85.$

Those values are near $2^8/3 = 85\frac{2}{3}$. Explore more examples, formulate some conjectures, and try to prove them.

(b) Are sums of every fourth entry in Row n close to the "expected" value $2^n/4$?

Are the sums of every fifth entry fairly close to $2^n/5$? Any conjectures?

Problem 6

Two rows $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ of n numbers (or symbols) are said to be *orthogonal* if:

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \;=\; 0.$$

For instance, rows $(1, 2, 3)$ and $(4, -5, 2)$ are orthogonal because $1\cdot 4 + 2\cdot(-5) + 3\cdot 2 = 4 - 10 + 6 = 0$. Similarly: Rows (a, b) and $(-b, a)$ are orthogonal for any a, b.

Suppose $(x_1, x_2, \dots, x_n)$ is a row containing n different symbols (independent variables). Define a *mixing* of that row to be a rearrangement of the entries, with a plus or minus sign assigned to each term. For instance, the row (x_1, x_2, x_3) has 48 different mixings (I think), including $(-x_2, x_3, -x_1)$ and $(x_1, -x_3, x_2)$.

Sometimes, mixings of a row can be orthogonal to one another. For example, when $n = 4$ the three rows

$$\begin{array}{l} (\;\;x_1,\;\;\; x_2,\;\;\; x_3,\;\;\; x_4), \\ (-x_2,\;\;\; x_1,\;\;\; x_4, -x_3), \\ (\;\;x_3,\;\;\; x_4, -x_1, -x_2), \end{array}$$

are mutually orthogonal. That is, each row is orthogonal to the other two.

- How many mixings of the row $(x_1, x_2, \dots, x_n)$ can be mutually orthogonal?

For example, if n is odd, no two mixings can be orthogonal. Does that claim have some simple proof?
How many mutually orthogonal mixings can there be when $n = 4, 6$, or 8?

What observations can you make? What sorts of questions about this situation would be interesting to investigate?

We hope you enjoyed working on these problems! Information about this summer mathematics program is available on the web at http://u.osu.edu/rossmath/. Your questions and comments can be emailed to ross@math.ohio-state.edu.