Chicago is presently under pressure to integrate its schools. The first year minimum minority population was not acceptable. We felt a need to do something before someone began to apply pressure, so we went to PUSH EXCEL and convinced them to offer enrichment for those identified minority students who did not get into the classes because of their SAT scores. They agreed and offered Saturday enrichment classes.
For the 1979-80 school year we had a similar situation: gifted students and lack of money. For the second year of our program, the University of Illinois at Chicago Circle and Chicago State University offered assistance. (Northeastern Illinois University was unable to continue.) The curriculum for the first year of the program was refined and expanded to include the use of computers.

In the original program design, it was our desire to accelerate these eighth year students in algebra and to continue their acceleration in high school eventually into Advanced Placement Calculus. After completing AP Calculus, but while still in high school, the students would attend appropriate mathematics classes at a university or college of their choice.

For the 1979-80 school year, we had a small group of students who were in a state of limbo. These were the students who were in the seventh year for the 1978-79 program. Now they had successfully completed algebra and completed elementary school mathematics. A geometry class was formed for this group at UICC. (Since we only include seventh year students for testing after 1979, this situation will not occur again.) The geometry class also used the computer and studied program-solving strategies in algebra and geometry.

An enrichment program open to all the participants in the Talent Search (and indeed open to the general public) was also developed in conjunction with the universities. This was a Saturday morning lecture series conducted at the Museum of Science and Industry. Leading mathematicians conducted self-contained lectures at a level of understanding of our talented students. Attendance averaged close to 100 people per lecture for the first two years.

Evaluation

At the end of the year the students were evaluated in a variety of ways. Of course the instructors had monitored the performance throughout the year, keeping records of the homework, special projects, and puzzles completed. Teacher-made tests were administered to each class. In addition, the Educational Testing Service Cooperative Mathematics Test in Algebra was administered during the last week of class. For the 1979 classes, 91.7 percent of the students scored in the 99.6 percentile. Over half, 50.7 percent, scored in the 90th percentile, and 83 percent of the total scored in at least the 75th percentile. The 1980 test results were equally impressive, with 51 percent scoring at or above the 90th percentile, and 77.6 scoring at or above the 75th percentile. There was no direct correlation between SAT score and the course grade. This can be accounted for in part by motivation.

Summary

To summarize my remarks, nothing is impossible if we cooperate with one another. This program is the result of the cooperation between a large urban school system and several universities. This cooperation was responsible for serving the very special needs of some of the world's most precious resources, gifted students - - hopefully, the leaders of tomorrow.

But it is also a lesson in faith. The situation I have described may appear at first glance as possible only in a large city; however, most school systems have some university, college, or two-year college within proximity to themselves. If you have faith, nothing is impossible.

WHAT MATHEMATICS FOR GIFTED YOUNG PEOPLE: THE PROBLEMS OF SELECTION OF CONTENT AND OF BRINGING ABOUT DEEP STUDENT INVOLVEMENT

Arnold E. Ross
The Ohio State University
Columbus, Ohio, U.S.A.

In attempting to provide worthwhile mathematical experience for the very young, one quickly recognizes the inadequacy of teacher's eloquence as a surrogate for students experience. In our efforts to provide appropriate mathematical experience for the student we became keenly aware of the special qualities of mind characteristic of the various stages of development of very young individuals. Also, we have found that is is the better part of wisdom to provide a means of sharing the newly gained experience by borrowing from our linguistic friends the technique of introducing language through usage (1).

The opportunities and the challenges presented by the special qualities of mind of the very young are often very surprising and ever thought-provoking. Let me discuss two pages of the classroom logbook of a class of twelve and thirteen year old children who came to us, in part, from the inner city (2).

This particular class already had some experience with Euler graphs. It was my intention to experiment with graphs that morning until the class would discover conditions for a graph to be a Euler graph. To my surprise, each time I drew a graph on the blackboard there developed keen competition as to who would draw first the required complete path if such would exist or who would declare this to be impossible. So successful was this competition both in speed and in unfailing accuracy that I could not get anyone in the class to take a searching look at the graph itself and seek the reason for success or failure. We sidestepped further discussion of graphs that day and proceeded to do a partition problem.

When we returned to the graph problem on another day we concentrated on determining the degrees of vertices, the number of vertices, the number of edges and the relations between these. After some discussion about the hazards presented by vertices of odd degree, the correct conjecture about the criteria for an Euler graph was produced by the class.

The class was not experienced enough to seek a proof of sufficiency. We gave the essential part of the argument by taking advantage of the constructive nature of the proof and of the fact that everyone in the class was throwing away each initial unsuccessful attempt to draw a desired path. We suggested that one child could build on each unsuccessful attempt by starting again from a vertex of the first crossed path by a suitable cut and by the pasting together of the resulting pieces. Everyone liked this construction as assuring success and we used in on a number of examples.

The lack of space will not permit a detailed account of other mathematical ideas which proved to be very accessible to the very, very young.

Let me conclude this part by disputing the prevalent misconception that there is magic of success in combining applications with relevant mathematical ideas when one deals with an audience which lacks appropriate phenomenological experience. I have found that in order to achieve a really effective blending of the two kinds of experience one must take equally great pains in developing appreciation of the subtleties of the concerns of the experimentalist as one is willing to take in making accessible the subtleties of mathematical concepts.

In a science fiction industrial democracy such as ours the quality of life, indeed our very survival depends upon our resources of developed significant talent.

Although any effort to encourage worthwhile interest in the very young is preferable to neglect, we must, I believe, give adequate attention to the task of nurturing carefully, intensively, and persistently individuals with a very high potential for achievement. We do this in sports and we do this in music.

In sports, winning in competition is the popularly accepted objective of training. In music the end result is performance judged through many subtleties and in its deeper aspects not involving competition among individuals. Even greater subtleties and greater variations of intellectual temperament must be recognized in the development of creative individuals in mathematics, in science, and in the related disciplines.

Thus, whatever is the range of mathematical ideas with which we are concerned at any given time, the effective education of a creative scientist (mathematician! engineer!) demands that we encourage the kind of involvement which develops the student's capacity to observe keenly, to ask astute questions and to recognize significant problems. This last is important in scientific education for two reasons. First, the progress of every science depends upon the capacity of its practitioners to ask penetrating questions and to identify important problems. Second, we believe that personal discovery is a vital part of the learning process for every individual eager to gain deep insight into his subject.

Selection and organization of subject matter depends very much upon the experience and the ultimate destination of the young audience. In my experiment (1973 -) with a two-year mathematics honor sequence at the university, we included as many ideas as possible which every young scientist (and mathematician) should master as early as possible. We began with an intensive and problem oriented discussion of vectors in the analytic geometry of two and three dimensions. We used this experience to make the student feel at home with complex numbers and their geometric applications. We moved on to the study of derivatives of vector as well as scalar functions of the scalar variable. This permitted us to introduce elementary differential geometry. Students' familiarity with complex numbers made it possible not to separate completely the study of real and the study of complex calculus (such separation is an unfortunate current practice, I believe). We tried not to be glib about limits but to emphasize approximations and error estimates. We did a great deal with integration before we turned to power series. We studied the geometrical (qualitative) theory of nonlinear differential equations and used computers, digital as well as analog, to investigate (after Euler) the behavior of trajectories in a two dimensional phase space. We did linear algebra reaching canonieal forms more directly and more quickly than is usual. We did multivariate calculus and some geometrical determinant theory in connection with that. We learned to be precise and concise, avoided abstraction for mere pleasure, and laid considerable emphasis on problem solving.

In the limited time alloted to me in this session I would like to discuss in greater detail our work in the summer with mathematically gifted pre-college students whose ages range from 15 to 18. The duration of this program has varied from eight weeks at The Ohio State University in the U.S.A., to four weeks at the Mathematisches Instut of the Heidelberg University in Germany, to three weeks in Bangalore, India, to two weeks at the Australian National University in Canberra (3). In all these programs Number Theory has been used as the basic vehicle for the development of the student's capacity for observation, invention, the use of language, and all those traits of character which constitute intellectual discipline. (See plates 1 and 2.)

This has been a happy choice since number theory not only abounds in deep but accessible mathematical ideas but it is rich in opportunities for acquiring the kind of experience which is vitally needed in science and technology.

The traditional use of problems develops, in some instances to a very high degree, the student's capacity to resolve difficult problems provided they are clearly formulated.

For many years we have used suitably designed problems to develop also (4) the student's capacity for observation, to encourage the spirit of adventure in conjecturing and to develop their staying power in the exploration of what have been to them (as well as to the historical originators) some very difficult questions. Abstract reasoning is a very important tool in creative work in all the sciences and in technology. To master the use of this tool it is important, I believe, to give the student an opportunity to participate in the process of generalization. The design of problems must respond to this need as well (5).

It has been our practice in the above mentioned summer programs to group the problems in each Problem Set under subtitles which are meant to emphasize dramatically the purpose which the problems are meant to serve. In our Ohio State University program participants are given 40 sets of problems with the total number of problems exceeding 400.

The rather demanding studies during the first summer create a significant momentum for further studies in mathematics, science, or engineering. Those of our program participants who are not ready to enter college and who return to us for the second and possibly the third summer are able, as a rule, to study interesting and useful mathematics and science taught in a stimulating manner by our accomplished colleagues.

Math. Inst. Schülerseminar – Praktikum – Zahlentheorie, Aufgabe 7, 12|9|78
Heidelberg

ARNOLD E. ROSS

<u>Exploration – Erforschung.</u>

<u>P1</u>. In <u>P10</u>, Aufgabe 6 we discovered that if u_1, u_2 are elements of U_m, if n_1 is the order of u_1 and n_2 is the order of u_2 <u>and if</u> $(n_1, n_2) = 1$, then $n_1 n_2$ is the order of $u_1 u_2$. What useful observation can you make if $(n_1, n_2) > 1$? Perhaps the following example may help: Consider U_{19}. 12 is of order 6; 16 is of order 9; 3 is of order 18; 18 is the l.c.m. of 6 and 9; $3 = 16 \cdot 12^3$ and the order of 16 is 9 and of 12^3 is 2. We note that $(9,2) = 1$.

<u>Prove or disprove and salvage if possible.</u>

<u>P2</u>. If $u \in U_m$, if n is the order of u and if a is a positive integer such that $u^a = 1$, then $n|a$.

<u>P3</u>. 1) $\mathbb{Z}_p$ is a field if p is a prime. 2) $(\mathbb{Z}_3[x])_{x^2+1}$ also is a field. Explain carefully.

<u>P4</u>. A polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $a_n \neq 0$, of degree n with coefficients $a_0, a_1, \ldots a_n$ in a field F (F may be $\mathbb{Z}_p, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$) cannot have more than n distinct roots. Hint: A field has no divisors of zero.

<u>P5</u>. If $u \in U_m$ and n is the order of u in U_m, then by <u>P3</u> Aufgabe 6 (Prof. Jahnke's lectures), $n | \varphi(m)$ and hence $u^{\varphi(m)} = 1$ in U_m. In particular $u^{p-1} = 1$ in U_p if p is a prime.

<u>P6</u>. $\mu(ab) = \mu(a)\mu(b)$. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}_p[x]$.

<u>Technique of Generalization – Die Kunst von Verallgemeinerung.</u>

<u>P7</u>. On the basis of your experience with $\mathbb{Z}$, develop the arithmetic in $\mathbb{Z}_3[x]$. Would a similar discussion yield the arithmetic of $\mathbb{Z}_p[x]$, where p is a positive prime in $\mathbb{Z}$? Would such a discussion yield the arithmetic of $\mathbb{Q}[x]$? Of $\mathbb{R}[x]$? Of $\mathbb{C}[x]$? Justify your assertions.

<u>Numerical Problems (Some Food for Thought).</u>

<u>P8</u>. Find the g.c.d. (f,g) of $f(x) = x^4 - 3x^3 + 2x^2 + 4x - 1$ and $g(x) = x^2 - 2x + 3$ in $\mathbb{Z}_5[x]$.

<u>P9</u>. Use the results in <u>P8</u> to find a solution $\{X, Y\}$ of the "Diophantine" equation

$$f(x)X(x) + g(x)Y(x) = (f(x), g(x))$$

in $\mathbb{Z}_5[x]$. Does our algorithm in $\mathbb{Z}$ apply here?

<u>P10</u>. Find the g.c.d. of $7 + 11i$ and $3 + 5i$ in $\mathbb{Z}[i]$.

<u>P11</u>. Out of the ring $\mathbb{Z}_3[x]$ construct a ring $(\mathbb{Z}_3[x])_{x^2+x+2}$ of remainders in division by $x^2 + x + 2$. Do this in a manner similar to that used to construct $\mathbb{Z}_m$ out of $\mathbb{Z}$. How many distinct elements are there in $(\mathbb{Z}_3[x])_{x^2+x+2}$? Calculate the number of units in $(\mathbb{Z}_3[x])_{x^2+x+2}$. Is the group of units cyclic? Is $\sqrt{-1}$ in our new "ring of remainders"? Is our ring a field? Justify your assertions.

<u>Counting techniques – Die Kunst von Rechnung.</u>

<u>P12</u>. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_e^{\alpha_e}$ be the canonical factorization of a positive integer n into positive primes in $\mathbb{Z}$. Use the formulas in <u>P18</u>, Aufgabe 2 to obtain a formula for $\varphi(n)$.

<u>P13</u>. Calculate the number of units in $(\mathbb{Z}[i])_{3+i}$ by any method. Could one generalize the results in <u>P12</u> and introduce a function $\varphi(a_1 + a_2 i)$ with $a_1, a_2 \in \mathbb{Z}$ which would give the number of units in the ring $(\mathbb{Z}[i])_{a_1 + a_2 i}$ of remainders with respect to the modulus $a_1 + a_2 i$?

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Heidelberg

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Prove or disprove and salvage if possible.

P1. a, b, a', b' are all positive and $\frac{a}{b} < \frac{a'}{b'} \Rightarrow \frac{a}{b} < \frac{a+a'}{b+b'} < \frac{a'}{b'}$. The fraction $\frac{a+a'}{b+b'}$ is called the mediant of $\frac{a}{b}$ and $\frac{a'}{b'}$. Represent these fractions geometrically. What is the relation between the vector representing the mediant and the vectors representing $\frac{a}{b}$ and $\frac{a'}{b'}$?

P2. If $\frac{a}{b}$ and $\frac{a'}{b'}$ are two consecutive entries in a Farey sequence then there exists no fraction between $\frac{a}{b}$ and $\frac{a'}{b'}$ with the denominator smaller than $b+b'$. If our Farey sequence is of order n, then $b+b'>n$. We recall also that $a'b-ab'=1$.

P3. If $\frac{a}{b}, \frac{a''}{b''}, \frac{a'}{b'}$ are three consecutive entries in a Farey sequence of order n, then

$$\frac{a''}{b''} = \frac{a+a'}{b+b'}.$$

P4. p is a positive prime in $\mathbb{Z}$. $\pi(x)$ a prime of degree n in $\mathbb{Z}_p[x] \Rightarrow \pi(y) \mid y^{p^n}-y \Rightarrow \pi(y)$ splits into linear factors in $F=(\mathbb{Z}_p[x])_{\pi(x)}$. How do we find all the roots of $\pi(y)$ in F? For this last see P23, Aufgabe 11 and P6, Aufgabe 10.

P5. If u is a generator of the group U_p of units in $\mathbb{Z}_p$ and p is a positive prime in $\mathbb{Z}$, then u is not a square in $\mathbb{Z}_p$.

P6. Let $a=pq+r$, $0 \le r < p$. Then $\left[\frac{2a}{p}\right]$ is even if $0 \le r < \frac{p}{2}$ and $\left[\frac{2a}{p}\right]$ is odd if $\frac{p}{2} \le r < p$. (Compare with P9, Aufgabe 8.)

P7. $f(n) = \sum_{\substack{d \mid n \\ d>0}} g(d)$. Then if $g(n)$ is a multiplicative arithmetic function so is $f(n)$. We say that $g(n)$ is multiplicative if $g(n_1 n_2) = g(n_1)g(n_2)$ whenever we have $(n_1, n_2)=1$.

Numerical Problems (Some Food for Thought).

P8. We note that $\sqrt{-1} \notin \mathbb{Z}_7$. Construct a finite field $F \supset \mathbb{Z}_7$ and such that $\sqrt{-1} \in F$. Explain.

P9. In P23, Aufgabe 11 compare the roots of $f(y) = y^3+2y+4$ with the powers x, x^5, x^{25} of x. Explain.

Exploration - Erforschung.

P10. We consider the fundamental point lattice

$x\mathfrak{a}_1 + y\mathfrak{a}_2$ with x, y in $\mathbb{Z}$.

Consider next two vectors

$\mathfrak{b}_1 = \overrightarrow{OP}$ and $\mathfrak{b}_2 = \overrightarrow{OQ}$

and the point lattice

$x\mathfrak{b}_1 + y\mathfrak{b}_2$ with x, y in $\mathbb{Z}$.

which $\mathfrak{b}_1$ and $\mathfrak{b}_2$ determine.

In Fig. 1 is this point lattice the same as our fundamental point lattice? What about Fig 2? Explain and generalize.

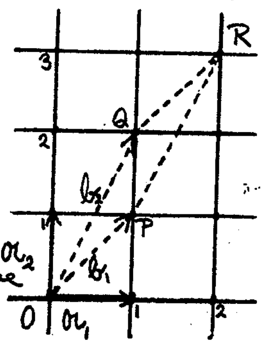

Fig. 1

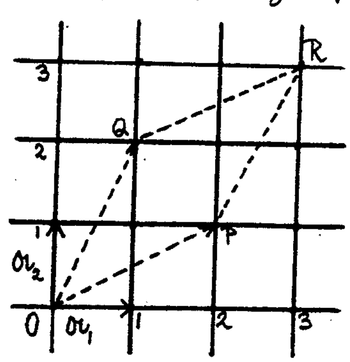

Fig. 2

P11. Let x, y be a solution of Pell's equation $x^2-dy^2=1$ in positive integers. We assume that $d>0$ and $d \ne \square$. Then

$$\left|\frac{x}{y}-\sqrt{d}\right| < \frac{1}{2y^2}.$$

Is $\frac{x}{y}$ a "better" approximation to $\sqrt{d}$ than you expected? What interesting conclusions can you draw?

The Ohio State University, Columbus, Ohio 43210, U.S.A.
1980 Summer Program.

Plate #2.

In the summer of 1980 we introduced a triple track course in analysis for our returnees. The elementary calculus of p-adic variable (Track 1) was taught by Professor Kurt Mahler and contained material from the second edition of his Cambridge Tract: On p-adic numbers and their functions. The basic real analysis (Track 2) was taught by Professor Bogdan Baishanski. A problem seminar pointing up the similarities and the differences between the properties of real valued and p-adic valued functions was directed by Professor Ranko Bojanic.

All of our advanced participants studied p-adic and real calculus. The more advanced among them also participated in one of the following three courses: A course in geometry taught by Professors Hans Zassenhaus and Jill Yaqub, a course in Algorithmic Methods taught by Professor Hans Zassenhaus, and a university course in Combinatorics taught by Professors Thomas Dowling and Neil Robertson was open to our advanced participants.

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GUIDANCE FOR TEACHERS IN THE IDENTIFICATION OF AND PROVISION FOR MATHEMATICALLY GIFTED CHILDREN

Graham T. Hoare
Dr. Challoner's Grammar School
Buckinghamshire, England

In a section concerning the provision of Primary and Secondary Education, the 1944 Education Act of England and Wales charged the local authorities with the responsibility of ensuring that their schools should provide education for all pupils according to 'their different ages, abilities and aptitudes'. It has become increasingly clear, however, that many with marked mathematical ability are insufficiently challenged and stretched in the classroom. In his fascinating book, 'Adventures of a Mathematician', Stanislaw Ulam, referring to mathematical talent, thinks 'there is an almost continuous passage from mediocrity to the highest levels of people like Gauss, Poincare and Hilbert'. Accordingly, below the potential genius level, we would expect to find a considerable reservoir of talent, and much evidence accrues to justify our belief in its existence. In a recent book of essays, entitled 'Mathematics Today', Allen Hammond refers to experimental programmes in which practicing mathematicians have taught very young ghetto children advanced algebra and similar subjects. The rapidity with which the students absorbed concepts far advanced beyond those normally taught at their grade level, their enthusiasm, and the rapport between them and mathematicians astounded professional educators. To the charge that, by showing particular concern for the gifted, we are creating an elite, we would reply that this minority already exists, whether we decide to ignore or frustrate it, or to provide appropriate facilities and opportunities for nurturing it.

Who are the gifted and how can we identify them? Although it is hard to quantify mathematical talent, an initial screening could be made by conducting, for example, a non-verbal reasoning group test at quite an early age. Choosing, administering and judging the purpose of tests, and interpreting the scores on them, require expertise which teachers generally do not have, and they would be strongly advised to seek the guidance of a trained psychologist in these tasks. We might add that we know of no written test which, unaided by skillful interviewing, could be used to isolate deep aspects or levels of mathematical understanding for individuals.

It is, however, classroom teachers who are the most important agents in the identification process and they need, therefore, to be aware of those characteristics which distinguish the mathematically able. The list below, which owes much to V.A. Krutetskii, can be used to guide and refine the teacher's subjective observations and assessments of individual children. By studying the processes, rather than the results, of mathematical thinking, Krutetskii's approach to the subject of mathematical ability differs markedly from that employed by factor analysts.

1. Facility for generalisation; verbally and symbolically.

2. Capacity and facility for abstract thinking.

3. Flexibility in thinking, ingenuity; inventiveness.

4. Striving for clarity, simplicity, conciseness.

5. Quickness in comprehending new ideas and the ability to apply them readily.

6. Logical acumen - a grasp of the notion of proof and of an algorithm.

7. Mathematical memory.

8. "Mathematical cast of mind" - a tendency to interpret the world mathematically.

These characteristics interweave and influence each other, although they may emerge and be recognised at different times in a child's development. Some of them may not be intrinsically mathematical qualities, but mathematically gifted pupils reveal them especially when dealing with numerical and spatial concepts and relationships. Number itself is an abstraction, and a facility for and a delight in handling numbers might be an early indication of mathematical talent, but care is needed to distinguish this from the more limited ability to do calculations.