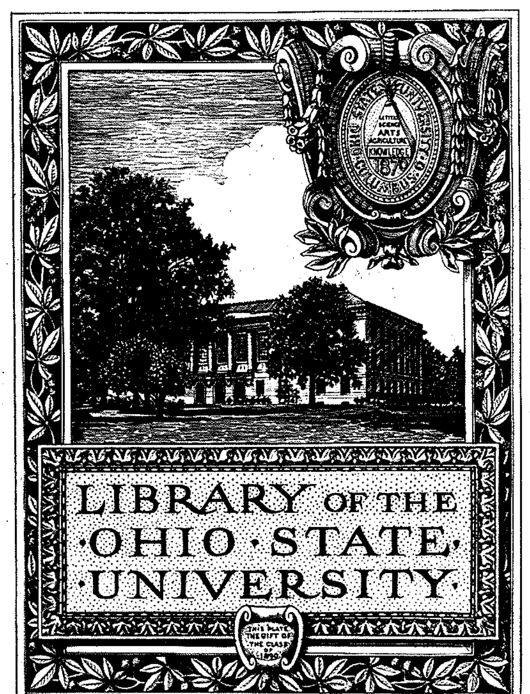

Applications of the Theory of Elliptic Functions to the Theory of Numbers

by

P. S. Nazimoff

Translated From Russian

By

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The original was published in Moscow in 1884

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PREFACE TO THE AMERICAN EDITION

In presenting this translation of the more original p[arts] of Nazimoff's remarkable book to those who do not read Russi[an], it is the hope of the editors that not only will the individ[ual] results of the work become more widely known, and therefore p[re]vent useless duplication, but also that the ingenious and powe[r]ful methods introduced by Nazimoff will pass into the common ar[s]enal of working arithmeticians. Outside of Russia, Nazimoff's work has had practically no influence on the development of tho[se] parts of arithmetic to which it is devoted. For this undeserved neglect the author himself is largely responsible, as his own French abstract in the Journal de l'Ecole Polytechnique does bu[t] scant justice to the book. The meagre abstract in the Fortschri[tte] is based on the French account, not on the Russian.

Certain of the topics treated here have inherited an augmented interest since Nazimoff wrote. His numerous theorems on numbers of representations in certain quadratic forms in the ear[l]ier chapters, for example, will be of great value to workers on what Dickson has called universal forms (those that represent al[l] integers, or all positive integers). Again, the arithmetical proof[s] for certain of Liouville's general formulas in the theory of numbers will doubtless stimulate research in these powerful methods. Although many of Nazimoff's isolated results have been obtained otherwise by later writers, there is a sufficient residue of generality and originality in each of the chapters to make all worth of close attention.

The theme of the book is arithmetic, particularly the broad discussion of what is attainable in the theory of quadratic forms by use of elliptic functions. Its spirit is algebraic. Some will prefer to use strictly elementary methods to reach these or similar results, but few will deny, from the evidence presented here, that Nazimoff's algebraic attack is one of great power in discovery, whatever means may be selected for the final presentation of the proofs. To all who prefer algebra and analysis to pure arithmetic as tools for exploration, this book will prove invaluable.

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The copy of Nazimoff's book from which this translation was made is probably the only one in the United States; it was given to me by Professor James V. Ouspensky at the Toronto Congress in 1924. To Professor L. E. Dickson, at whose instigation the translation was undertaken, and to his pupils who have made the translation and prepared the stencils, all friends of arithmetic owe their thanks for rendering Nazimoff's work at last accessible to American and English readers.

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TRANSLATOR'S PREFACE

This translation was undertaken by me at the suggestion of Prof. L.E. Dickson and Prof. E.T. Bell and was originally intended for the private use of a few of their students. The work was done at the close of my senior year at the University in what spare time I had. It was later decided to mimieograph the translation for use in Prof. Bell's class during the coming summer and offer it to the interested ones outside of the univeristy. It is very unfortunate that neither Prof. Dickson nor Prof. Bell had the opportunity to read the manuscript before stencils were cut.

Mr. O. E. Brown, who cut the stencils, has read each page once but more thorough proof reading could not be arranged. It is his plan to distribute a list of corrections later and he hopes that the reader will note and report all mistakes found.

In his introduction (which was not translated) Nazimoff introduces the following notations which he employs throughout the book:

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \qquad K' \int_0^1 \frac{dx}{\sqrt{(1-x^2)[1-(1-k^2)x^2]}},$$

$$q = e^{-\frac{\pi K'}{K}}, \qquad k' = \sqrt{1-k^2}$$

$$\vartheta(x,q) = \vartheta(x) = 1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + \dots,$$

$$\vartheta_3(x,q) = \vartheta_3(x) = 1 + 2q\cos 2x + 2q^4\cos 4x + 2q^9\cos 6x + \dots,$$

$$\vartheta_1(x,q) = \vartheta_1(x) = 2q^{\frac{1}{4}}\sin x - 2q^{\frac{9}{4}}\sin 3x + 2q^{\frac{25}{4}}\sin 5x - 2q^{\frac{49}{4}}\sin 7x + \dots,$$

$$\vartheta_2(x,q) = \vartheta_2(x) = 2q^{\frac{1}{4}}\cos x + 2q^{\frac{9}{4}}\cos 3x + 2q^{\frac{25}{4}}\cos 5x + 2q^{\frac{49}{4}}\cos 7x + \dots,$$

$$\sin \operatorname{am}\left(\frac{2Kx}{\pi}, q\right) = \sin \operatorname{am}\left(\frac{2Kx}{\pi}, k\right) = \sin \operatorname{am}\frac{2Kx}{\pi} = \frac{1}{\sqrt{k}}\,\frac{\vartheta_1(x)}{\vartheta(x)},$$

$$\cos \operatorname{am}\left(\frac{2Kx}{\pi}, q\right) = \cos \operatorname{am}\left(\frac{2Kx}{\pi}, k\right) = \cos \operatorname{am}\frac{2Kx}{\pi} = \sqrt{\frac{k'}{k}}\,\frac{\vartheta_2(x)}{\vartheta(x)},$$

$$\Delta \operatorname{am}\left(\frac{2Kx}{\pi}, q\right) = \Delta \operatorname{am}\left(\frac{2Kx}{\pi}, k\right) = \Delta \operatorname{am}\frac{2Kx}{\pi} = \frac{\sqrt{k'}\,\vartheta_3(x)}{\vartheta(x)}.$$

Arnold Chaimovitch

University of Chicago,
May 1928.