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Towards the Abstract

ARNOLD E. ROSS
Ohio State University

1. Our plan

We propose to consider a very basic kind of human experience, namely, the experience of dealing with *sets* (classes) of objects. We shall show that out of the study of such simple surroundings mathematics grows by the process of observation, experimentation, discovery and invention.

We shall construct a kind of algebra whose elements are sets (of objects) and we shall study in detail the properties of such an algebra. We shall then introduce other examples of algebraic systems, and we shall propose a way of comparing such systems. We shall discover that mathematical systems can resemble each other in some fundamental way, and we shall be led to a discussion of the process of abstraction (the algebra of sets, the algebra of divisors and the algebra of logic, all leading to the abstract system called Boolean algebra).

2. Some basic questions concerning sets

In each discussion we select a set I of objects which we agree to consider and which we call our *universe of discourse*. As the elements of our mathematical system we shall take the *subsets* of our universe of discourse.

We say that a set A is a *subset* of a set B or that A is contained in B if and only if every element of A is an element of B. This does not exclude the case when $A = B$. If A is contained in B we write $A \subset B$. If p is an element of A we write $p \in A$. We indicate implication (if, then) by the double arrow $\Rightarrow$ and equivalence (if and only if) or twofold implication by $\Leftrightarrow$. Thus, in shorthand,

$$A \subset B \Leftrightarrow (p \in A \Rightarrow p \in B).$$

A set may be described by exhibiting its elements or the names of these elements. Thus we may describe the set of all students in the freshman mathematics class by referring to the full list I, where I consists of all students in our college. If the universe of discourse I is taken to be the set of all integers, we may speak of the set

$$A = \{1, 2, 5, 10\},$$

or the set

$$B = \{1, 3, 7, 9\}.$$

A set may also be described by indicating the characteristic properties of its elements, i.e. the properties which are possessed by all the elements of the given set and by no other elements. Thus the last two sets may be described as follows:

$A = \{\text{all positive integral divisors of } 10\}$,
$B = \{\text{all } x \text{ such that } x \text{ is integral, } 1 \leqslant x \leqslant 10, x \text{ is relatively prime to } 10\}$,

or, in shorthand,

$A = \{x\!: x \text{ positive integer, } x \mid 10\}$,
$B = \{x\!: x \text{ integral, } 1 \leqslant x \leqslant 10, (x, 10) = 1\}$.

Here $\{x\!: x \text{ has property } P\}$ is read as *all* x such that x has the property P.

Problem 1. The following sets are described in terms of the characteristic properties of their elements. Describe these sets by exhibiting their elements. In each case a suitable universe of discourse is presupposed.

$A = \{x\!: x \text{ a prime number, } 1 < x < 50\}$,
$B = \{z\!: z \text{ a complex number, } z^4 = 1\}$,
$C = \{\text{all right triangles with integral sides } m, n, k \text{ and with the area } A < 200\}$,
$D = \{c\!: c \text{ a coefficient in the expansion of } (x + y)^{11}\}$.

Problem 2. Describe the following sets by giving the characteristic properties of their elements:
$A = \{1, 2, 3, 5, 6, 10, 15, 30\}$,
$B = \{1, 7, 11, 13, 17, 19, 23, 29\}$,
$C = \{1, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i\}$,
$D = \{6, 28, 496, 8128\}$.

Given two sets A and B (in our chosen universe of discourse I) it is natural to inquire about the set of elements which A and B have in common (the *intersection* of A and B) and also about the totality of all elements contained in at least one of the two sets A, B (the *union* of A and B). We denote the intersection of A and B by $A \cap B$ and their union by $A \cup B$. The complement of a set A is the set A' of all the elements in I but not in A. In shorthand,

$D = A \cap B \Leftrightarrow (p \in D \Leftrightarrow p \in A \text{ and } p \in B)$,
$U = A \cup B \Leftrightarrow (p \in U \Leftrightarrow p \in A \text{ or } p \in B)$,
$C = A' \Leftrightarrow (p \in C \Leftrightarrow p \notin A) \Rightarrow A' \cup A = I$.

We should note that the disjunction 'or' is used by mathematicians in the sense of 'one or the other *or both*.' In common parlance the last alternative is most often excluded. The inquiry 'Are you a man or a mouse?' excludes the possibility of both alternatives holding true simultaneously.

3. The algebra of sets

We are well on the way to constructing our mathematical system. We have on hand the elements of our system. These are the subsets of our universe of discourse I. The operations of taking the union and the intersection of two sets remind one of the familiar binary operations of arithmetic. However, we still have some subtle adjustments to make before we can achieve our end.

We recall that a *binary operation* in a set S is given by a rule which assigns a *unique element* of S to *every pair* of elements of S. We see that the taking of the union determines a binary operation in S, where S is the *set of all the subsets of I* having at least one element. Taking the intersection of two sets A and B in S yields another set in S only when A and B have elements in common. To provide two disjoint sets with an intersection we introduce a fictitious subset $\varnothing$ called the *null set* (popularly referred to as the set without elements) and write $A \cap B = \varnothing$ if A and B are disjoint. With this convention the taking of the intersection is a binary operation in the set $S = \mathscr{P}(I)$ of all subsets of I *including the null subset* $\varnothing$. The real mathematical meaning of the null set will become clearer as our discussion develops. We shall find it convenient to take $A \cup \varnothing = A = \varnothing \cup A$ for every $A \in \mathscr{P}(I)$. Then the union is also a binary operation in $\mathscr{P}(I)$.

If we agree that $\varnothing' = I$ and $I' = \varnothing$, then taking the complement of a set is a *unary operation* in $\mathscr{P}(I)$, i.e. to *every element* A of $\mathscr{P}(I)$ it assigns a *unique element* of $\mathscr{P}(I)$.

4. Conjecture, counterexample, proof

Problem 3. Let $I = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{1, 3, 5, 7\}$, $D = \{7, 8, 9, 10\}$, $E = \{1, 2, 3, 4, 5, 6\}$. What sets are determined by the following algebraic expressions: $(A \cup B) \cap C$, $(A \cap C) \cup (B \cap C)$, $A \cup B$, $B \cup A$, $A \cap C$, $(A \cap D) \cup C$, $C \cap A$, $(B \cap C) \cup A$, $(B \cup A) \cap (C \cup A)$, $A' \cap B'$, $(A \cup B)'$, $C' \cup D'$, $(A \cap B) \cap C$, $(C \cap D)'$, $A \cap (B \cap C)$, $B \cup (C \cup D)$, $B \cup E$, $(B \cup C) \cup D$, $E \cap E$? Here the parentheses indicate, as usual, the order in which the operations are to be carried out.

Would the reader venture any guesses regarding the properties of the operations of our new algebra on the basis of the above calculations?

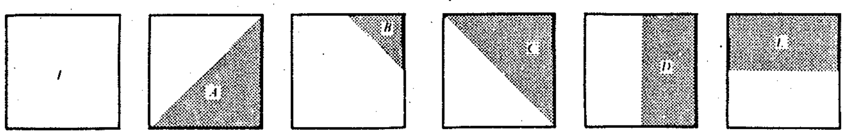

Figure 1

Problem 4. Let I consist of all points in the left-hand square in Figure 1 and let A, B, C, D, E be the subsets of I indicated by the shading. In a separate figure for each expression indicate by shading the sets determined by each algebraic expression given in Problem 3. Do the results of the new 'calculations' give additional support to the guesses suggested by the results in Problem 3?

We observe that in Problem 3, $E \cap B = \{3, 4, 5\} = B$. Also in Problem 4 $E \cap B = B$. We conjecture that $X \cap Y = Y$ for every pair of sets X and Y. That is, we conjecture that the last equality is *an identity*. However, substituting the sets A and B in Problem 3 for X and Y respectively we find that $A \cap B = \{3\} \neq B$. Thus we have found an example which is contrary to our conjecture (*a counterexample*) and have shown through this that $X \cap Y = Y$ is not an identity.

We see that in order to disprove the truth of a universal assertion we need to find but one counterexample.

It does not suffice, however, to find a number of instances for which our assertion holds true in order to be sure that it holds universally. Consider the equality (12)† $X' \cup Y' = (X \cap Y)'$ for example. This equality is verified by every pair of sets in Problem 3. This fact does not serve as a guarantee that greater perseverance will not produce a counterexample. To show that (12) is indeed an identity we must resort to a detailed analysis of its meaning.

To arrange our discussion in a neat manner we observe that the very definition of the equality of two sets as sets having exactly the same elements may be written in the form

$$U = V \Leftrightarrow [(p \in U \Rightarrow p \in V) \text{ and } (p \in V \Rightarrow p \in U)]$$

which means that

$$U = V \Leftrightarrow U \subset V \text{ and } U \supset V. \qquad (*)$$

We next observe that our basic definitions yield the following chain of implications for every X and Y:

$$p \in X' \cup Y' \Leftrightarrow p \in X' \text{ or } p \in Y' \Leftrightarrow p \notin X \text{ or } p \notin Y \Leftrightarrow p \notin X \cap Y \Leftrightarrow p \in (X \cap Y)'.$$

Reading this argument from left to right we see that $X' \cup Y' \subset (X \cap Y)'$. Reading from right to left shows that $X' \cup Y' \supset (X \cap Y)'$. That (12) is an identity follows then from $(*)$.

Problem 5. Which of the following equalities are identical equalities (identities)? In each case either find a counterexample or give an 'elementwise' proof such as the one given for (12).†‡

(1) $X \cap X = X$, (2) $X \cup Y = Y \cup X$, (3) $(X \cup Y)' = X' \cup Y'$,
(4) $X \cap (Y \cap Z) = (X \cap Y) \cap Z$, (5) $(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z)$,
(6) $\varnothing \cap X = \varnothing$, (7) $I \cup X = I$, (8) $X \cup X' = I$, (9) $X \cap X' = \varnothing$,
(10) $X \cup Y = Y$, (11) $X \cup (Y \cap Z) = (X \cup Y) \cap Z$, (12) $(X \cap Y)' = X' \cup Y'$,
(13) $(X \cup Y)' = X' \cap Y'$, (14) $X \cap (X \cup Y) = X$,
(15) $(X' \cup Y')' \cup (X' \cup Y)' = X$, (16) $X \cup (X \cap Y) = X$, (17) $X \cup X = X$,
(18) $X \cap Y = Y \cap X$, (19) $X \cup (Y \cup Z) = (X \cup Y) \cup Z$,
(20) $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$, (21) $\varnothing \cup X = X$, (22) $I \cap X = X$,
(23) $(X')' = X$.

† The numbers in round brackets here and below refer to the lists in Problems 5 and 6.
‡ We suggest that at a first reading, it is sufficient to look in detail at (say) the odd-numbered examples only.

Problem 6. Which of the following statements hold true universally?

(24) $X \cap Y = X \Leftrightarrow X \subset Y \Leftrightarrow X \cup Y = Y$, (25) $X \subset X$,
(26) $X \subset Y$ and $Y \subset Z \Rightarrow X \subset Z$, (27) $\varnothing \subset X$, (28) $X \subset I$,
(29) $A \cap X = A \cap Y$ and $A \cup X = A \cup Y \Rightarrow X = Y$, (30) $X \subset X \cup Y$,
(31) $X \supset X \cap Y$, (32) $T = (X \cap T) \cup (X' \cap T) \Leftrightarrow X = \varnothing$,
(33) $X \subset A' \Leftrightarrow X \cap A = \varnothing$, (34) $Y \supset A' \Leftrightarrow A \cup Y = I$,
(35) $X \cap Y' = Z \cap Z' \Leftrightarrow X \cap Y = X$, (36) $X \subset Y \Leftrightarrow X' \supset Y'$,
(37) $X \subset Y$ and $X \subset Z \Rightarrow X \subset Y \cap Z$, (38) $X \subset Z$ and $Y \subset Z \Rightarrow X \cup Y \subset Z$,
(39) $X = Y \Leftrightarrow X \subset Y$ and $Y \subset X$, (40) $X \subset Y \Leftrightarrow X' \cup Y = I$,
(41) $X \subset Y \Rightarrow X \cup Z \subset Y \cup Z$, (42) $X \subset Y \Rightarrow X \cap Z \subset Y \cap Z$.

The student should test his powers of observation and his capacity for initiative on the examples in Problems 3–6. He should go through the various stages involved in the process of mathematical discovery, viz. 1) experimentation and observation, 2) making a conjecture, 3) testing for possible counterexamples, 4) justification.

5. Interdependence of properties of a mathematical system

Problem 7. Show that for every two sets X and Y in I, the four sets $X \cap Y$, $X \cap Y'$, $X' \cap Y'$, $X' \cap Y$ form a partition of I, i.e. these sets are pairwise disjoint (which means that no two of them have elements in common) and their union is I.

It can be shown directly without resorting to an elementwise argument that some universal statements are consequences of one or more other universal statements. Thus (1) is a consequence of (31), (37) and (39). On the other hand (29) can be deduced from (31), (18), (24), (2) and (5).

Problem 8. Justify the steps in the above derivation of (1) and (29).

We shall speak of this type of derivation as a relative argument, to distinguish it from the elementwise method of proof. The relative argument brings out (indeed is based on) the pattern of dependence which exists among various true statements in a given mathematical system. It is the study of these patterns that a mathematician has in mind when he speaks of the study of the structure of such systems.

6. The game of a 'reduced inventory': the first step towards abstraction

Let us list all the statements in Problems 5 and 6 (or, say, the odd-numbered ones) which can be proved to hold true by an elementwise argument. Then let us play the following game. We shall allow ourselves to drop a statement from this list if its truth follows by a relative argument from a number of other true statements in our listing. From what is left we can again drop a true statement provided it is derivable from others which have not yet been dropped from the list, and so on. This process will terminate either when the remaining statements can no longer be derived from each other or through the limitations of the player's ingenuity.

We may, as an example, start our game as follows: We drop (29) from our initial list since it follows from (31), (24), (2), (5) and (18). Since (1) is a consequence of (31),

(37) and (39), none of which has been dropped, we may drop (1) in addition to (29). Next, we note that (8) follows from (25), (2) and (40), all of which still remain in our listing. Hence (8) can also be dropped.

Proceeding in this manner we can go as far as our ingenuity will take us.

We should note that if every statement in the reduced list which is left at the end of the game is proved by an elementwise argument, then all other statements on the original list can be deduced from those on the reduced list by the usually less cumbersome relative argument.

Different players are likely to arrive at different though equally useful reduced lists. Birkhoff and MacLane (reference 1) use (1), (17), (2), (18), (4), (19), (5), (20), (24), (7), (28), (21), (22), (6), (27), (8), (9), (12), (13), (23). Rosenbloom (reference 3) essentially uses (18), (4), (35) and Courant and Robbins (reference 2) suggest (2), (19), (15).

A relative argument involving only the identities in a reduced list of statements is called *algebraic manipulation.*

Problem 9. Show using algebraic manipulation that

$$(A \cap B)' = (A \cap B') \cup (A' \cap B) \cup (A' \cap B') \text{ is an identity.}$$

Proof. $(A \cap B)' = A' \cup B' = (A' \cap I) \cup (B' \cap I) = [(A' \cap B) \cup (A \cap B')] \cup [(B' \cap A) \cup (B' \cap A')] = (A \cap B') \cup (A' \cap B) \cup (A' \cap B')$.

Justify each step of the above proof by references to appropriate identities in Problem 5. Comment on the identity just proved in the light of the partition described in Problem 7.

7. Systems with identical reduced inventories of properties: the second step toward abstraction

Let us now make what will seem, at first glance, to be a digression but which will prove to be the promised thought-provoking surprise of our discussion.

Problem 10. Show that the taking of the greatest common divisor (a, b) of two integers a and b is a binary operation in the set S if 1) S is the set of positive integers; 2) $S = S_n$ is the *set of all positive divisors* of $n = 6$, or of $n = 12$, or of $n = 30$ or of an arbitrary integer n. Show that for each of the above sets S, the taking of the lowest common multiple $[a, b]$ of two integers a and b is also a binary operation.

Problem 11. To conform with the usual custom of placing the symbol for binary operations between the two elements involved, we shall write $a \wedge b = (a, b) =$ g.c.d. of a and b, and $a \vee b = [a, b] =$ l.c.m. of a and b. If a divides c, we write $a < c$. Show that the following statements hold true in S_6, S_{12}, S_n (n an integer) and S:

(1) $x \wedge x = x$, (2) $x \vee y = y \vee x$, (4) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, (5) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$, (20) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$, (24) $x \wedge y = x \Leftrightarrow x < y \Leftrightarrow x \vee y = y$. Observe the similarity with the like numbered statements in Problems 5 and 6.

What identities in our present systems correspond to (21) and (22) in Problem 5? In other words, what elements in our system behave like $\varnothing$ and I in the algebra of sets? In what way is this behaviour the same as that of zero relative to addition and unity relative to multiplication in ordinary arithmetic in the set of all integers $\mathbf{Z}$?

What statements in Problems 5 and 6 have a bearing on the conditions

$$a \wedge x = 1,\ a \vee x = n (\text{in } S_n)? \qquad (**)$$

Does (**) have a solution x for every $a \in S_{12}$? Show that (**) does have a solution x for every $a \in S_6$. Show that the same is true for S_{30}. Can one generalize this? In view of this discussion, how will you define a' in S_6, in S_{30}?

Show (the second step toward abstraction) that all the statements involving $\wedge$, $\vee$, $<$, and $'$ in S_6 (also in S_{30}) which correspond to true statements in Problems 5 and 6 are also true. For what integers n can this be said of S_n? (The answer is given at the end of the article.) Note that in order to prove our assertion *it suffices to compare corresponding reduced systems.*

If we do not inquire into the nature of the elements (sets of objects in Problems 5 and 6, positive divisors of an integer n in the example above) but are concerned only with the properties of binary or unary operations and of some binary relations such as $\subset$ or $<$, and if these have the same properties as the operations and relations in our two examples, viz., the algebra of sets and the 'algebra of divisors', which we introduced, then we say that we study Boolean† algebras and that we deal with an abstract theory of which our examples serve as *realizations.*

$\wedge$	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

$\vee$	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

a	a'
1	6
2	3
3	2
6	1

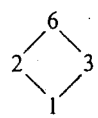

Figure 2

It is said that Boolean algebras are obtained by the process of generalization. The question arises as to how general is 'general'. It is interesting to note that for each Boolean algebra of a finite number of elements one can obtain a realization as an algebra of sets which gives a *true (faithful) representation* of our Boolean algebra. We shall try to convey this subtle idea by means of an example.

To bring out clearly the meaning of what we have in mind, we make use of the familiar representation of operations through the use of tables, in the same way in which multiplication in arithmetic is represented by the multiplication tables.

Thus, the tables for $\wedge$, $\vee$, the table for $'$ and the charts for $<$ in the case of S_6 are given in Figure 2.

† After the English logician George Boole (1815–1864).

On the other hand, the tables for $\cap$, $\cup$, the table for complementation, and the chart for $\subset$ for $I = \{1, 2\}$ are given in Figure 3.

$\cap$	$\varnothing$	$\{1\}$	$\{2\}$	$\{1, 2\}$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
$\{1\}$	$\varnothing$	$\{1\}$	$\varnothing$	$\{1\}$
$\{2\}$	$\varnothing$	$\varnothing$	$\{2\}$	$\{2\}$
$\{1, 2\}$	$\varnothing$	$\{1\}$	$\{2\}$	$\{1, 2\}$

$\cup$	$\varnothing$	$\{1\}$	$\{2\}$	$\{1, 2\}$
$\varnothing$	$\varnothing$	$\{1\}$	$\{2\}$	$\{1, 2\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1, 2\}$	$\{1, 2\}$
$\{2\}$	$\{2\}$	$\{1, 2\}$	$\{2\}$	$\{1, 2\}$
$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$

A	A'
$\varnothing$	$\{1, 2\}$
$\{1\}$	$\{2\}$
$\{2\}$	$\{1\}$
$\{1, 2\}$	$\varnothing$

Figure 3

We notice that the two systems are 'identical except for notation'. More precisely we can establish a 'one-to-one' correspondence $[1 \leftrightarrow \varnothing, 2 \leftrightarrow \{1\}, 3 \leftrightarrow \{2\}, 6 \leftrightarrow I]$ so that if $A \leftrightarrow a$, $B \leftrightarrow b$, then $A \cap B \leftrightarrow a \wedge b$, $A \cup B \leftrightarrow a \vee b$, $A' \leftrightarrow a'$, $A \subset B \Leftrightarrow a < b$. Hence the corresponding statements in the two systems either both hold true or both are false.

Problem 12. Show that the Boolean algebra of the positive divisors of 30 (the case S_{30}) is faithfully represented by the algebra of subsets of $I = \{1, 2, 3\}$.

We say that two systems which are faithful replicas of each other represent one and the same *abstract* mathematical system.

Note that the algebra of sets with $I = \{1, 2, 3\}$ is *essentially different* from the algebra of positive divisors of 6, *in the sense of not being faithful replicas of each other*.

Other realizations of Boolean algebras are used in logic and, through that, in the design of the high-speed digital computers (reference 4). It is primarily this last application that shifted the position of Boolean algebras away from 'pure mathematics' to make it one of the more popular subjects in 'applied mathematics'.

(The answer to the question on p. 94 is that n must contain no repeated factor.)

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