Ross Program 2023 Application Problems

This document is one part of the application to the *Ross Mathematics Program*, and will remain posted at https://rossprogram.org/students/to-apply from January through March.

The deadline for applications is March 31, 2023. The Admissions Committee will start reading applications in April.

Work independently on the problems below. We are interested in seeing how you approach and explore unfamiliar open-ended math problems, not whether you can find answers by searching through web sites or books, or by asking other people.

Submit your own work on these problems.

For each problem, explore the situation (with calculations, tables, pictures, etc.), observe patterns, make some guesses, test the truth of those guesses, and write logical proofs when possible. Where were you led by your experimenting?

Include your thoughts (but not your scratch-paper) even if you might not have found a complete solution. If you've seen one of the problems before (e.g. in a class or online), please include a reference along with your solution.

We are not looking for quick answers written in minimal space. Instead, we hope to see evidence of your explorations, conjectures, proofs, and generalizations written in a readable format.

The quality of mathematical exposition, the questions you pose, as well as the correctness and completeness of your solutions to those questions, are factors in admission decisions.

PDF format is required.

You may type your solutions using LaTeX or with a word processor, and then convert the output to PDF format.

Alternatively, you may scan your solutions from a handwritten paper copy, and convert that file to PDF. (Use dark pencil or pen and write on only one side of the paper.) Submitting photos of your work is not recommended since file sizes of photos are often too large. (The Ross system cannot accept files much larger than 5 megabytes.) Rather than photographs, you might use a "scan" feature on your camera.

Problem 1

Let $\mathbb{Z}$ denote the set of integers. If m is a positive integer, we write $\mathbb{Z}_m$ for the system of "integers modulo m." Some authors write $\mathbb{Z}/m\mathbb{Z}$ for that system.

> For completeness, we include some definitions here. The system $\mathbb{Z}_m$ can be represented as the set $\{0, 1, \ldots, m-1\}$ with operations $\oplus$ (addition) and $\odot$ (multiplication) defined as follows. If a, b are elements of $\{0, 1, \ldots, m-1\}$, define:
>
> $a \oplus b =$ the element c of $\{0, 1, \ldots, m-1\}$ such that $a + b - c$ is an integer multiple of m.
>
> $a \odot b =$ the element d of $\{0, 1, \ldots, m-1\}$ such that $ab - d$ is an integer multiple of m.
>
> For example, $3 \oplus 4 = 2$ in $\mathbb{Z}_5$, $3 \odot 3 = 1$ in $\mathbb{Z}_4$, and $-1 = 12$ in $\mathbb{Z}_{13}$.
> To simplify notations (at the expense of possible confusion), we abandon that new notation and write $a + b$ and ab for the operations in $\mathbb{Z}_m$, rather than writing $a \oplus b$ and $a \odot b$.

Let $\mathbb{Q}$ denote the system of rational numbers.

We write $4\mathbb{Z}$ for the set of multiples of 4 in $\mathbb{Z}$. Similarly for $4\mathbb{Z}_{12}$.

Consider the following number systems:

$$\mathbb{Z}, \quad \mathbb{Q}, \quad 4\mathbb{Z}, \quad \mathbb{Z}_3, \quad \mathbb{Z}_8, \quad \mathbb{Z}_9, \quad 4\mathbb{Z}_{12}, \quad \mathbb{Z}_{13}.$$

One system may be viewed as *similar* to another in several different ways. We will measure similarity using only algebraic properties.

(a) Consider the following sample properties:

- (i) If $a^2 = 1$, then $a = \pm 1$.
- (ii) If $2x = 0$, then $x = 0$.
- (iii) If $c^2 = 0$, then $c = 0$.

Which of the systems above have properties (i), (ii), and/or (iii)?

(b) Formulate another algebraic property and determine which of those systems have that property. [Note: Cardinality is not considered to be an algebraic property.]

Write down some additional algebraic properties and investigate them.

(c) In your opinion, which of the listed systems are "most similar" to each another?

Problem 2

Robot Rossie moves within a square room $ABCD$. Rossie moves along straight line segments, never leaving that room.

When Rossie encounters a wall she stops, makes a right-angle turn (with direction chosen to face into the room), and continues in that new direction.

If Rossie comes to one of the room's corners, she rotates through two right angles, and moves back along her previous path.

Suppose Rossie starts at point P on AB and her path begins as a line segment of slope s.
We hope to describe Rossie's path.

For some values of P and s, Rossie's path will be a tilted rectangle with one vertex on each wall of the room. (Often, this inscribed rectangle is itself a square.) In this case, Rossie repeatedly traces that *stable rectangle.*

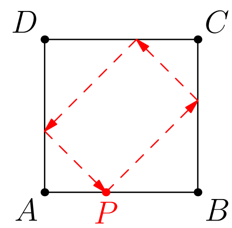

Figure 1: Stable rectangle when $s = 1$.

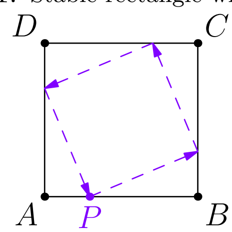

Figure 2: Stable rectangle when $0 < s < 1$.

(a) Suppose $s = 1$ so that the path begins at a 45 degree angle.
For every starting point P, show: Rossie's path is a stable rectangle.
(If P is a corner point, the path degenerates to a line segment traced back and forth.)

Now draw some examples with various P and s.
Given P and s, does Rossie's path always converge to a stable rectangle?
Here are some steps that might help you answer this question:

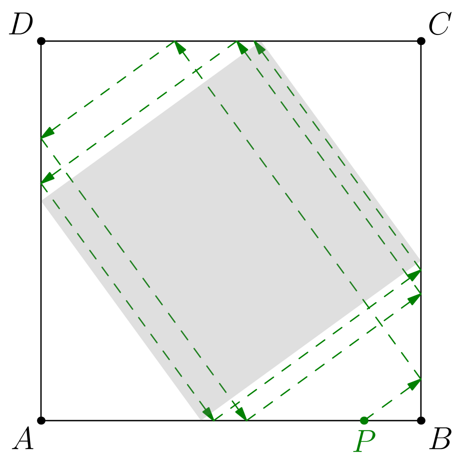

Figure 3: A stable rectangle for $0 < s < 1$ is shaded. Rossie's path seems to approach that rectangle.

(b) First consider the case: $0 < s < 1$. For each such s, is there exactly one stable rectangle with slope s?
Must that rectangle be a square?

(c) Suppose s is given with $0 < s < 1$. When Rossie starts at point X, let XX' be the first segment of her path. If X is on side AB, must X' be on BC? For P, Q on AB, how is the length $|P'Q'|$ related to $|PQ|$ and s?

(d) When $0 < s < 1$, prove or disprove:
For any starting point P, Rossie's path converges to a stable rectangle.

(e) What is Rossie's behavior when $s > 1$ or when $s < 0$?
Does the argument above still apply?

Problem 3

Please spend extra effort to write up this problem's solution as an exposition that can be read and understood by a beginning algebra student. That student knows function notation and standard properties of polynomials (as taught in a high school algebra course). Your solution will be graded not only on the correctness of the math but also on the clarity of exposition.

(a) Find all polynomials f that satisfy the equation:

$$f(x+2) = f(x) + 2 \quad \text{for every real number } x.$$

(b) Find all polynomials g that satisfy the equation:

$$g(2x) = 2g(x) \quad \text{for every real number } x.$$

(c) The problems above are of the following type: Given functions H and J, find all polynomials Q that satisfy the equation:

$$J\big(Q(x)\big) = Q\big(H(x)\big) \quad \text{for every } x \text{ in } S$$

where S is a subset of real numbers. In parts (a) and (b), we have $J = H$ and S is all real numbers, but other scenarios are also interesting. For example, the choice $J(x) = 1/(x-1)$ and $H(x) = 1/(x+1)$, generates the question:

Find all polynomials Q that satisfy the equation:

$$\frac{1}{Q(x)-1} = Q\left(\frac{1}{x+1}\right)$$

for every real number x such that those denominators are nonzero.

Is this one straightforward to solve?

(d) Make your own choice for J and H, formulate the problem, and find a solution. Choose J and H to be non-trivial, but still simple enough to allow you to make good progress toward a solution.