Ross Program 2020 Application Problems

This document is part of the application to the *Ross Mathematics Program*, and will remain posted at https://rossprogram.org/students/to-apply from January through March.

The Admission Committee will make acceptance decisions on a rolling basis, starting in March 2020. The deadline for applications is April 1, but spaces will fill as applications arrive. For adequate consideration of your application, it is best to **submit your solutions well before the end of March.**

Work independently on the problems below. We are interested in seeing how you approach unfamiliar math problems, not whether you can find answers by searching through web sites or books, or by asking other people.

Please submit your own work on these problems.

For each problem, explore the situation (with calculations, tables, pictures, etc.), observe patterns, make some guesses, test the truth of those guesses, and write logical proofs when possible. Where were you led by your experimenting?

Include your thoughts (but not your scratch-paper) even if you might not have found a complete solution. If you've seen one of the problems before (e.g. in a class or online), please include a reference along with your solution.

We are not looking for quick answers written in minimal space. Instead, we hope to see evidence of your explorations, conjectures, and proofs written in a readable format.

> The quality of mathematical exposition, as well as the correctness and completeness of your solutions, are factors in admission decisions.

Please convert your problem solutions into a PDF file. You may type the solutions using LaTeX or a word processor, and convert the output to PDF format.
Alternatively, you may scan your solutions from a handwritten paper copy, and convert the output to PDF. (Please use dark pencil or pen and write on only one side of the paper.) Submitting photos of your work is possible but not recommended: The resulting PDF files are often large, and the writing can be blurry and difficult to read.

Note: Unlike the problems here, each Ross Program course concentrates deeply on one subject. These problems are intended to assess your general mathematical background and interests.

Problem 1

Suppose $A = (a_n) = (a_1, a_2, a_3, \dots)$ is an increasing sequence of positive integers. A number c is called *A-expressible* if c is the alternating sum of a finite subsequence of A. To form such a sum, choose a finite subset of the sequence A, list those numbers in increasing order (no repetitions allowed), and combine them with alternating plus and minus signs. We allow the trivial case of one-element subsequences, so that each a_n is A-expressible.

Definition. Sequence $A = (a_n)$ is an "alt-basis" if every positive integer is uniquely A-expressible. That is, for every integer $m > 0$, there is exactly one way to express m as an alternating sum of a finite subsequence of A.

Examples. Sequence $B = (2^{n-1}) = (1,\ 2,\ 4,\ 8,\ 16, \dots)$ is not an alt-basis because some numbers are B-expressible in more than one way. For instance $3 = -1 + 4 = 1 - 2 + 4$.

Sequence $C = (3^{n-1}) = (1,\ 3,\ 9,\ 27,\ 81, \dots)$ is not an alt-basis because some numbers (like 4 and 5) are not C-expressible.

(a) Let $D = (2^n - 1) = (1,\ 3,\ 7,\ 15,\ 31, \dots)$. Note that:

$$\mathbf{1} = 1, \quad \mathbf{2} = -1+3, \quad \mathbf{3} = 3, \quad \mathbf{4} = -3+7, \quad \mathbf{5} = 1-3+7,$$
$$\mathbf{6} = -1+7, \quad \mathbf{7} = 7, \quad \mathbf{8} = -7+15, \quad \mathbf{9} = 1-7+15, \quad \dots$$

Prove that D is an alt-basis.

(b) Can some $E = (2, 3, \dots)$ be an alt-basis? That is, can you construct an alt-basis $E = (e_n)$ with $e_1 = 2$ and $e_2 = 3$?

(c) Can some $F = (1, 4, \dots)$ be an alt-basis? Justify your answer.

(d) Investigate some other examples. Is there some fairly simple test to determine whether a given sequence $A = (a_n)$ is an alt-basis?

Problem 2

A polynomial $f(x)$ has the *factor-square property* (or FSP) if $f(x)$ is a factor of $f(x^2)$. For instance, $g(x) = x - 1$ and $h(x) = x$ have FSP, but $k(x) = x + 2$ does not.

Reason: $x - 1$ is a factor of $x^2 - 1$, and x is a factor of x^2, but $x + 2$ is not a factor of $x^2 + 2$.

Multiplying by a nonzero constant "preserves" FSP, so we restrict attention to polynomials that are *monic* (i.e., have 1 as highest-degree coefficient).

What patterns do monic FSP polynomials satisfy?

To make progress on this topic, investigate the following questions and justify your answers.

(a) Are x and $x - 1$ the only monic FSP polynomials of degree 1?

(b) List all the monic FSP polynomials of degree 2.
To start, note that x^2, $x^2 - 1$, $x^2 - x$, and $x^2 + x + 1$ are on that list.
Some of them are products of FSP polynomials of smaller degree. For instance, x^2 and $x^2 - x$ arise from degree 1 cases. However, $x^2 - 1$ and $x^2 + x + 1$ are new, not expressible as a product of two smaller FSP polynomials.
Which terms in your list of degree 2 examples are new?

(c) List all the monic FSP polynomials of degree 3. Which of those are new?

Can you make a similar list in degree 4?

(d) Answers to the previous questions may depend on what coefficients are allowed. List the monic FSP polynomials of degree 3 that have integer coefficients. Separately list those (if any) with complex number coefficients that are not all integers.
Can you make similar lists for degree 4?
Are there examples of monic FSP polynomials with real number coefficients that are not all integers?

Problem 3

Here we work in the system of integer polynomials. Those are polynomials of the form $f(x) = r_n x^n + \cdots + r_1 x + r_0$ where every coefficient r_j is an integer.

General question:
When does some combination of the polynomials $ax + b$ and $cx + d$ equal 1 ?
That is, when do there exist integer polynomials $P(x)$ and $Q(x)$ with

$$P(x)\cdot(ax + b) + Q(x)\cdot(cx + d) = 1 \ ?$$

We concentrate here on cases when $c = 0$.

(a) Prove: No combination of $2x + 5$ and 3 can equal 1.
That is, no integer polynomials $P(x), Q(x)$ can satisfy:

$$P(x)\cdot\big(2x + 5\big) + Q(x)\cdot\big(3\big) = 1.$$

(b) Find a combination of $2x + 5$ and 4 that equals 1.

(c) Does some combination of $15x+9$ and 25 equal 1 ? How about $15x+9$ and 20 ?
Explain your reasoning.

(d) Investigate further examples of $ax + b$ and d, deciding in each case whether 1 is a combination. What patterns do you detect?
Can you prove that some of your observed patterns always hold true?

Problem 4

Let $S = \{1, 2, 3, \ldots, n\}$, a set of n elements. Suppose $\diamond$ is a binary operation S. Then $\diamond$ combines two elements $x, y \in S$ to yield an element $x \diamond y \in S$. We consider various properties that such an operation might have (like being associative).

That operation $\diamond$ is called *unital* if

$$1 \diamond x = x \diamond 1 = x, \quad \text{for every } x \in S.$$

The operation $\diamond$ is called *sandwiching* if

$$(x \diamond y) \diamond (z \diamond w) = (x \diamond z) \diamond (y \diamond w), \quad \text{for every } x, y, z, w \in S.$$

(a) If $\diamond$ is unital and sandwiching, must it be commutative?
That is, must $\diamond$ satisfy: $x \diamond y = y \diamond x$ for every $x, y \in S$?

(b) Must a unital, sandwiching operation $\diamond$ be associative?
That is, must $\diamond$ satisfy: $x \diamond (y \diamond z) = (x \diamond y) \diamond z$ for every $x, y, z \in S$?

(c) An operation $\bullet$ is called *self-distributive* if:

$$x \bullet (y \bullet z) = (x \bullet y) \bullet (x \bullet z), \quad \text{for every } x, y, z \in S.$$

Must a unital, self-distributive operation $\bullet$ be associative?

(d) We mentioned five properties above:
unital, sandwiching, commutative, associative, and self-distributive.
Do those properties satisfy some other implications? For example, if an operation $\diamond$ is self-distributive, must it be associative? Must it be unital?

Invent some other "natural" properties that a binary operation might have. What relationships can you discover among the axioms you have invented, and the five listed above?